CHAPTER 2

INPUT AND OUTPUT CHARACTERISTICS

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1 INTRODUCTION

Everyone is familiar with the interaction of devices connected to form a system, although they may not think of their observations in those terms. Familiar examples include the following:

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1. Dimming of the headlights while starting a car
2. Slowdown of an electric mixer lowered into heavy batter
3. Freezing a showerer by starting the dishwasher
4. Speedup of a vacuum cleaner when the hose plugs
5. Two-minute wait for a fever thermometer to rise
6. Special connectors required for TV antennas
7. Speedup of a fan in the window with the wind against it
8. Shifting of an automatic transmission on a hill

These effects happen because one part of a system loads another. Most mechanical engineers would guess that weighing an automobile by placing a bathroom-type scale under its wheels one at a time and summing the four measurements will yield a higher result than would be obtained if the scale was flush with the floor. Most electrical engineers understand that loading a potentiometer's wiper with too low a resistance makes its dial nonlinear for voltage division. Instrumentation engineers know that a heavy accelerometer mounted on a thin panel will not measure the true natural frequencies of the panel. Audiophiles are aware that loudspeaker impedances must be matched to amplifier impedance. We have all seen the 75- and 300-Ω markings under the antenna connections on TV sets, and most cable subscribers have seen balun transformers for connecting a coaxial cable to the flat-lead terminals of an older TV.

Every one of these examples involves a desired or undesirable interaction between a source and a receiver of energy. In every case, there are properties of the source part and the load part of the system that determine the efficiency of the interaction. This chapter deals exclusively with interactions between static and dynamic subsystems intended to function together in a task and with how best to configure and characterize those subsystems.

Consider the analysis of dynamic systems. To create mathematical models of these systems requires that we idealize our view of the physical world. First, the system must be identified and separated from its environment. The environment of a system is the universe outside the free body, control volume, or isolated circuit. The combination of these, which is the system under study and the external sources, provides or removes energy from the system in a known way. Next, in the system itself, we must arrange a restricted set of ideal elements connected in a way that will correctly represent the energy storages and dissipations of the physical system while, at the same time, we need the mathematical handles that explore the system's behavior in its environment. The external environment of the system being modeled must then itself be modeled and connected and is usually represented by special ideal elements called sources.

We expect, as a result of these sources, that the system under study will not alter the important variables in its environment. The water rushing from a kitchen faucet will not normally alter the atmospheric pressure; our electric circuit will not measurably slow the turbines in the local power plant; the penstock will not draw down the level of the reservoir (in a time frame consistent with a study of penstock dynamics, anyway); the cantilever beam will not distort the wall it is built into; and so on. In this last instance, for example, the wall is a special source of zero displacement and zero rotation no matter what forces and moments are applied.

In this chapter, we consider, instead of the behavior of a single system in a known environment, the interaction between pairs of connected dynamic systems at their interface, often called the driving point. The fundamental currency is, as always, the energy or power exchanged through the interface. In an instrumentation or control system, the objective of

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the energy exchange might be information transmission, but this is not considered here (we would like information exchanges to take place at the lowest possible energy costs, but the second law of thermodynamics rules out a free transmission).

As always, energy factors into two variables, such as voltage and current in electrical systems, and we are concerned with the behavior of these in the energetic interaction. The major difference in this perspective is that the system supplying energy cannot do so at a fixed value. Neither the source nor the system receiving energy can fix its values for a changing demand without a change in the value of a supply variable. The two subsystems are in an equilibrium with each other and are forced by their connection to have the same value of both of the appropriate energy variables. We concern ourselves with determining and controlling the value of these energy variables at the interface where, obviously, only one is determined by each of the interacting systems.

2 FAMILIAR EXAMPLES* OF INPUT–OUTPUT INTERACTIONS

2.1 Power Exchange

In the real world, pure sources and sinks are difficult to find. They are idealized, convenient constructs or approximations that give our system analyses independent forcing functions. We commonly think of an automobile storage battery as a source of 12.6 V independent of the needed current, and yet we have all observed dimming headlights while starting an engine. Clearly, the voltage of this battery is a function of the current demanded by its load. Similarly, we cannot charge the battery unless our alternator provides more than 12.6 V, and the charging rate depends on the overvoltage supplied. Thus, when the current demanded or supplied to a battery approaches its limits, we must consider that the battery really looks like an ideal 12.6-V source in series with a small resistance. The voltage at the battery terminals is a function of the current demanded and is not independent of the system loading or charging it in the interaction. This small internal resistance is termed the output impedance (or input impedance or driving-point impedance) of the battery.

If we measure the voltage on this battery with a voltmeter, we should draw so little current that the voltage we see is truly the source voltage without any loss in the internal resistance. The power delivered from the battery to the voltmeter is negligible (but not zero) because the current is so small. Alternatively, if we do a short-circuit test of the battery, its terminal voltage should fall to zero while we measure the very large current that results. Again, the power delivered to the ammeter is negligible because, although the current is very large, the voltage is vanishingly small.

At these two extremes the power delivered is essentially zero, so clearly at some intermediate load the power delivered will be a maximum. We will show later that this occurs when the load resistance is equal to the internal resistance of the battery (a point at which batteries are usually not designed to operate). The discussion above illustrates a simple concept: Impedances should be matched to maximize power or energy transfer but should be maximally mismatched for making a measurement without loading the system in which the measurement is to be made. We will return to the details of this statement later.

*Many of the examples in this chapter are drawn from Chapter 6 of a manuscript of unpublished notes, "Dynamic Systems and Measurements," by C. L. Nachtigal, used in the School of Mechanical Engineering, Purdue University, 1978.

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2.2 Energy Exchange

Interactions between systems are not restricted to resistive behavior, nor is the concept of impedance matching restricted to real, as opposed to reactive, impedances. Consider a pair of billiard balls on a frictionless table (to avoid the complexities of spin), and consider that their impact is governed by a coefficient of restitution, ϵ. Before impact, only one ball is moving, but afterward both may be. The initial and final energies are as follows:

$$\text{Initial energy} = \tfrac{1}{2}M_1 v_{1i}^2$$

$$\text{Final energy} = \tfrac{1}{2}M_1 v_{1f}^2 + \tfrac{1}{2}M_2 v_{2f}^2 \tag{1}$$

where the subscript 1 refers to the striker and 2 to the struck ball, M is mass, v is velocity, and the subscripts i and f refer to initial and final conditions, respectively.

Since no external forces act on this system of two balls during their interaction, the total momentum of the system is conserved:

$$M_1 v_{1i} + M_2 v_{2i} = M_1 v_{1f} + M_2 v_{2f}$$

or

$$v_{1i} + \mathbf{m} v_{2i} = v_{1f} + \mathbf{m} v_{2f} \tag{2}$$

where $\mathbf{m} = M_2/M_1$. The second equation, required to solve for the final velocities, derives from impulse and momentum considerations for the balls considered one at a time. Since no external forces act on either ball during their interaction except those exerted by the other ball, the impulses,* or integrals of the force acting over the time of interaction on the two, are equal. (See *impact* in virtually any dynamics text.) From this it can be shown that the initial and final velocities must be related:

$$\epsilon(v_{1i} - v_{2i}) = (v_{1f} - v_{2f}) \tag{3}$$

where $v_{2i} = 0$ in this case and the coefficient of restitution ϵ is a number between 0 and 1. A 0 corresponds to a plastic impact while a 1 corresponds to a perfectly elastic impact. Equations (2) and (3) can be solved for the final velocities of the two balls:

$$v_{1f} = \frac{1 - \mathbf{m}\epsilon}{1 + \mathbf{m}} v_{1i} \qquad \text{and} \qquad v_{2f} = \frac{1 + \epsilon}{1 + \mathbf{m}} v_{1i} \tag{4}$$

Now assume that one ball strikes the other squarely† and that the coefficient of restitution ϵ is unity (perfectly elastic impact). Consider three cases:

1. The two balls have equal mass, so $\mathbf{m} = 1$, and $\epsilon = 1$. Then the striking ball, M_1, will stop, and the struck ball, M_2, will move away from the impact with exactly the initial velocity of the striking ball. *All the initial energy is transferred.*
2. The struck ball is more massive than the striking ball, $\mathbf{m} > 1$, $\epsilon = 1$. Then the striker will rebound along its initial path, and the struck ball will move away with less than the initial velocity of the striker. *The initial energy is shared between the balls.*

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*Impulse $= \int_{t=0}^{t}$ Force dt, where Force is the vector sum of all the forces acting over the period of interaction, t.

†Referred to in dynamics as *direct central impact.*

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3. The striker is the more massive of the two, $\mathbf{m} < 1$, $\epsilon = 1$. Then the striker, M_1, will follow at reduced velocity behind the struck ball after their impact, and the struck ball will move away faster than the initial velocity of the striker (because it has less mass). Again, *the initial energy is shared between the balls.*

Thus, the initial energy is conserved in all of these transactions. But the energy can be transferred completely from one ball to the other *if and only if* the two balls have the same mass.

If these balls were made of clay so that the impact was perfectly plastic (no rebound whatsoever), then $\epsilon = 0$, so the striker and struck balls would move off together at the same velocity after impact no matter what the masses of the two balls. They would be effectively stuck together. The final momentum of the pair would equal the initial momentum of the striker because, on a frictionless surface, there are no external forces acting, but energy could not be conserved because of the losses in plastic deformation during the impact. The final velocities for the same three cases are

$$v_f = \frac{1}{1 + \mathbf{m}} v_i \tag{5}$$

Since the task at hand, however, is to transfer kinetic (KE) from the first ball to the second, we are interested in maximizing the energy in the second ball after impact with respect to the energy in the first ball before impact:

$$\frac{\mathrm{KE}_{(M_2,\mathrm{after})}}{\mathrm{KE}_{(M_1,\mathrm{before})}} = \frac{\frac{1}{2}M_2(v_{2f})^2}{\frac{1}{2}M_1(v_{1i})^2} = \frac{M_2(1/(1 + \mathbf{m}))^2(v_{1i})^2}{M_1(v_{1i})^2} = \frac{\mathbf{m}}{(1 + \mathbf{m})^2} \tag{6}$$

This takes on a maximum value of $\frac{1}{4}$ when $\mathbf{m} = 1$ and falls off rapidly as $\mathbf{m}$ departs from 1.

Thus, after the impact of two clay balls of equal mass, one-fourth of the initial energy remains in the striker, one-fourth is transferred to the struck ball, and one-half of the initial energy of the striker is lost in the impact. If the struck ball is either larger or smaller than the striker, however, then a greater fraction of the initial energy is dissipated in the impact and a smaller fraction is transferred to the second ball. The reader should reflect on how this influences the severity of automobile accidents between vehicles of different sizes.

2.3 A Human Example

Those in good health can try the following experiment. Run up a long flight of stairs one at a time and record the elapsed time. After a rest, try again, but run the stairs two at a time. Still later, try a third time, but run three steps at a time. Most runners will find that their best time is recorded for two steps at a time.

In the first test, the runner's legs are velocity limited: Too much work is expended simply moving legs and feet, and the forces required are too low to use the full power of the legs effectively. In the third test, although the runner's legs do not have to move very quickly, they are on the upper edge of their force capabilities for continued three-step jumps; the forces required are too high and the runner could, at lower forces, move his or her legs much faster. In the intermediate case there is a match between the task and the force–velocity characteristics of the runner's legs.

Bicycle riders assure this match with a variable-speed transmission that they adjust so they can crank the pedals at approximately 60 RPM. We will later look at other means of ensuring the match between source capabilities and load requirements when neither of them

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is changeable, but the answer is always a transformer or gyrator of some type (a gear ratio in this case).

3 ENERGY, POWER, IMPEDANCE

3.1 Definitions and Analogies

Energy is the fundamental currency in the interactions between elements of a physical system no matter how the elements are defined. In engineering systems, it is convenient to describe these transactions in terms of a complementary pair of variables whose product is the power or flow rate of the energy in the transaction. These product pairs are familiar to all engineers: voltage × current = power, force × displacement = energy, torque × angular velocity = power, pressure × flow = power, and pressure × time rate of change of volume exchanged = power. Some are less familiar: flux linkage × current = energy, charge × voltage = energy, and absolute temperature × entropy flux = thermal power. Henry M. Paynter's[1] *tetrahedron of state* shows how these are related (Fig. 1). Typically, one of these factors is *extensive,* a flux or *flow,* such as current, velocity, volume flow rate, or angular velocity. The other is *intensive,* a potential or *effort,** such as voltage, force, pressure, or torque. Thus $\mathcal{P}$ = *extensive* × *intensive* for any of these domains of physical activity.

This factoring is quite independent of the analogies between the factors of power in different domains, for which any arbitrary selection is acceptable. In essence, velocity is not like voltage or force like current, just as velocity is not like current or force like voltage. It is convenient, however, before defining impedance and working with it to choose an analogy so that generalizations can be made across the domains of engineering activity. There are two standard ways to do this: the *Firestone* analogy[2] and the *mobility* analogy. Electrical engineers are most familiar with the Firestone analogy, while mechanical engineers are probably more comfortable with the mobility analogy. The results derived in this chapter are independent of the analogy chosen. To avoid confusion, both will be introduced, but only the mobility analogy will be used in this chapter.

The Firestone analogy gives circuitlike properties to mechanical systems: All systems consist of nodes like a circuit and only of lumped elements considered to be two-terminal or four-terminal devices. For masses and tanks of liquid, one of the terminals must be understood to be ground, the inertial reference frame, or atmosphere. Then one of the energy

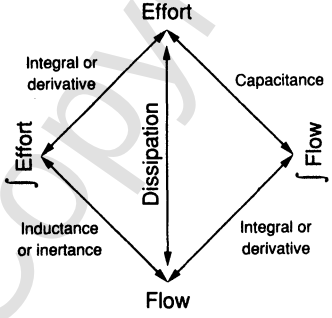


Figure 1 H. M. Paynter's *tetrahedron of state.*

*This is Paynter's terminology, used with reference to his "Bond Graphs."

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variables is measured *across* the terminals of the element and the other passes *through* the element. In a circuit, voltage is across and current passes through. For a spring, however, velocity difference is *across* and the force passes through. Thus this analogy linked voltage to velocity, angular velocity, and pressure as across variables and linked current to force, torque, and flow rate as through variables. Clearly, across × through = power.

The mobility analogy, in contrast, considers the complementary power variables to consist of a potential and a flux, an intrinsic and extrinsic variable. The potentials, or *efforts,* are voltage, force, torque, and pressure, while the fluxes, or *flows,* are current, velocity, angular velocity, and fluid flow rate.

3.2 Impedance and Admittance

Impedance, in the most general sense, is the relationship between the factors of power. Because only the constitutive relationships for the dissipative elements are expressed directly in terms of the power variables, $\Delta V_R = R \cdot i_R$ for example, while the equations for the energy storage elements are expressed in terms of the derivative of one of the power variables* with respect to the other, $i_C = C(dV_C/dt)$ for example, these are most conveniently expressed in Laplace transform terms. Impedances are really self-transfer functions at a point in a system. Since the concept was probably defined first for electrical systems, that definition is most standardized: Electrical impedance $Z_{\text{electrical}}$ is defined as the rate of change of voltage with current:

$$Z_{\text{electrical}} = \frac{d(\text{voltage})}{d(\text{current})} = \frac{d(\text{effort})}{d(\text{flow})} \tag{7}$$

By analogy, impedance can be similarly defined for the other engineering domains:

$$Z_{\text{translation}} = \frac{d(\text{force})}{d(\text{velocity})} \tag{8}$$

$$Z_{\text{rotation}} = \frac{d(\text{torque})}{d(\text{angular velocity})} \tag{9}$$

$$Z_{\text{fluid}} = \frac{d(\text{pressure})}{d(\text{flow rate})} \tag{10}$$

Table 1 is an impedance table using these definitions of the fundamental lumped linear elements. Note that these are derived from the Laplace transforms of the constitutive equa-

Table 1 Impedances of Lumped Linear Elements

Domain	Kinetic Storage	Dissipation	Potential Storage
Translational	Mass: Ms	Damping: b	Spring: k/s
Rotational	Inertia: Js	Damping: B	Torsion spring: k_t/s
Electrical	Inductance: Ls	Resistance: R	Capacitance: $1/Cs$
Fluid	Inertance: Is	Fluid resistance: R	Fluid capacitance: $1/Cs$

*See Fig. 1 again. Capacitance is a relationship between the integral of the flow and the effort, which is the same as saying that capacitance relates the flow to the derivative of the effort.

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tions for these elements; they are the transfer functions of the elements and are expressed in terms of the Laplace operator s. The familiar $F = M \cdot a$, for example, becomes, in power-variable terms, $F = M(dv/dt)$; it transforms as $F(s) = Msv(s)$, so

$$(Z_{\text{translation}})_{\text{mass}} = \frac{dF_{\text{mass}}}{dv_{\text{mass}}} = Ms \tag{11}$$

Because these involve the Laplace operator s, they can be manipulated algebraically to derive combined impedances. The reciprocal of the impedance, the *admittance,* is also useful. Formally, admittance is defined as

$$\text{Admittance: } Y = \frac{1}{Z} = \frac{d(\text{flow})}{d(\text{effort})} \tag{12}$$

3.3 Combining Impedances and/or Admittances

Elements in series are those for which the flow variable is common to both elements and the efforts sum. Elements in parallel are those for which the effort variable is common to both elements and the flows sum. By analogy to electrical resistors, we can deduce that the impedance sum for series elements and the admittance sum for parallel elements form the combined impedance or admittance of the elements:

Impedances *in series:*

$$\begin{aligned} &Z_1 + Z_2 = Z_{\text{total}} \quad \text{(common flow)} \\ &\frac{1}{Y_1} + \frac{1}{Y_2} = \frac{1}{Y_{\text{total}}} \end{aligned} \tag{13}$$

Impedances *in parallel:*

$$\begin{aligned} &Y_1 + Y_2 = Y_{\text{total}} \quad \text{(common effort)} \\ &\frac{1}{Z_1} + \frac{1}{Z_2} = \frac{1}{Z_{\text{total}}} \end{aligned} \tag{14}$$

When applying these relationships to electrical or fluid elements, there is rarely any confusion about what constitutes series and parallel. In the mobility analogy, however, a pair of springs connected end to end are in parallel *because they experience a common force,* regardless of the topological appearance, whereas springs connected side by side are in series *because they experience a common velocity difference.** For a pair of springs *end to end,* the total admittance is

$$\frac{s}{k_{\text{total}}} = \frac{s}{k_1} + \frac{s}{k_2}$$

so the impedance is

$$\frac{k_{\text{total}}}{s} = \frac{k_1 k_2}{s(k_1 + k_2)}$$

For the same springs *side by side,* the total impedance is

*For many, the appeal of the Firestone analogy is that springs are equivalent to inductors, and there can be no ambiguity about series and parallel connections. End to end is series.

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$$\frac{k_{\text{total}}}{s} = \frac{k_1 + k_2}{s}$$

3.4 Computing Impedance or Admittance at an Input or Output

There are basically two ways in which an input or output admittance can be computed. The first, and most direct, is to compute the transfer function between the effort and the flow at the driving point and take the derivative with respect to the flow. For a mechanical rotational system, for example, torque as a function of angular velocity is expressed and differentiated with respect to angular velocity. This method must be used if the system being considered is nonlinear because the derivative must be taken at an operating point. If the system is linear, then the ratio of flow or effort will suffice; in the rotational system, the impedance is simply the ratio τ/ω (torque/speed).

The second method takes the impedances of the elements one at a time and combines them. This approach is particularly useful for linear (or linearized) systems. The question then arises of determining the impedance of any sources in the subsystem being considered. Flow sources, such as current sources, velocity sources, angular velocity sources, and fluid flow sources, all have the relationship flow = constant. Their impedance is therefore infinite ($Z_{\text{flow source}} = \infty$) because any change in effort results in zero change in flow. Effort sources, such as voltage sources, force sources, torque sources, and pressure sources, will provide any flow to maintain the effort required; the change in effort for a change in flow remains zero, so their impedance is $Z_{\text{effort source}} = 0$. An effort source therefore represents a short circuit from an impedance point of view; it connects together two nodes that were separate. A flow source represents a null element; since its impedance is infinite, it represents an open circuit. Flow sources are simply removed in impedance calculations.

An example will distinguish between these two approaches. Figure 2 shows, on the left, a simple circuit disconnected at a driving point from its load. The load is of no consequence in this calculation; we simply require the driving-point impedance of the circuit. In the first approach, we derive the voltage at the driving point as a function of the current leaving those terminals and the *source*, V_s and I_s to obtain the relationship

$$V_o = \frac{R_3}{R_1 + R_3} V_s + \left(\frac{R_1R_2 + R_2R_3 + R_1R_3}{R_1 + R_3}\right)(I_s - i_o) \tag{15}$$

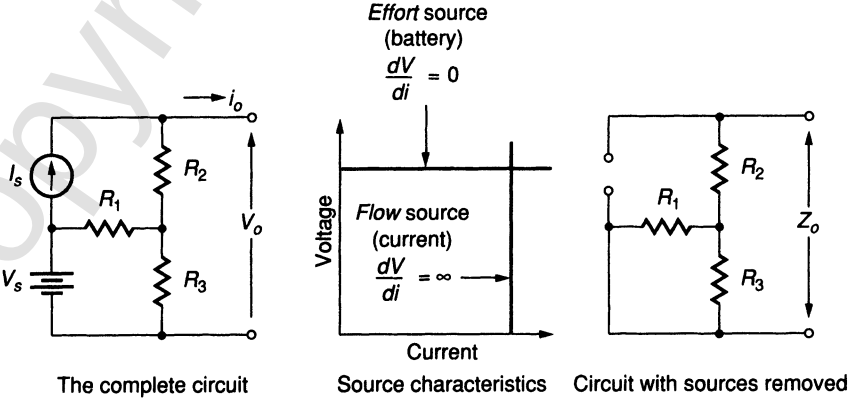


Figure 2 A simple circuit as a source.

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Clearly, the negative derivative of the voltage (V_o) with respect to the current (i_o) is given by

$$-\frac{dV_o}{di_o} = Z_o = \frac{R_1R_2 + R_2R_3 + R_1R_3}{R_1 + R_3} \tag{16}$$

In the second method, the voltage source (V_s) can be set to zero, a short circuit, without affecting the impedances, and the current source (I_s) can be removed to yield the circuit on the right in Fig. 2. Then the impedances need only be combined as R_2 in series with the parallel pair R_1 and R_3. Thus,

$$Z_o = R_2 + \frac{R_1R_3}{R_1 + R_3} = \frac{R_1R_2 + R_2R_3 + R_1R_3}{R_1 + R_3} \quad \text{as before} \tag{17}$$

3.5 Transforming or Gyrating Impedances

Ideal transformers, transducers, and gyrators play an important part in dynamic systems and an equally important part in obtaining optimal performance from source–load combinations. They share several vital features: All are two-port (or four-terminal) devices, all are energetically conservative, and all are considered lumped elements. Each of the many types requires two equations for its description, always of the same form. Table 2 lists many of the more common linear two-ports, and Fig. 3 illustrates them.

All of these transducing devices alter the *effort–flow* relationships of elements connected at their far end. Consider Fig. 4, which shows a simple model of a front wheel of an automobile suspension. Because the spring and damper are mounted inboard of the wheel, their effectiveness is reduced by the mechanical disadvantage of the suspension arm. What is the impedance of the spring and shock absorber at $(\cdot)_1$ as viewed from the wheel at $(\cdot)_2$?

Since the spring and shock absorber share a common velocity (both ends share nodes or points of common velocity), their impedances add to give the impedance of the pair at their point of attachment, $(\cdot)_1$, to the lever:

$$(Z_1)_{\text{total}} = Z_{\text{spring}} + Z_{\text{damper}} = b + \frac{k}{s} = \frac{bs + k}{s} \tag{18}$$

At the $(\cdot)_2$ end of the lever, the force, from Table 2, is $F_2 = (L_1/L_2)F_1$, but from the definition of Z for linear elements, $F_1 = (Z_1)_{\text{total}}v_1$. So we get the following:

$$F_2 = \frac{L_1}{L_2}\left(\frac{bs + k}{s}\right)v_1 \tag{19}$$

The second equation for the lever is $v_1 = (L_1/L_2)v_2$, and substituting this into Eq. (19) yields

$$F_2 = \frac{L_1}{L_2}\left(\frac{bs + k}{s}\right)\left(\frac{L_1}{L_2}\right)v_2 = \left(\frac{L_1}{L_2}\right)^2\left(\frac{bs + k}{s}\right)v_2 \tag{20}$$

$$(Z_2)_{\text{total}} = \left(\frac{L_1}{L_2}\right)^2 (Z_1)_{\text{total}} \tag{21}$$

Thus, the impedance of the suspension, observed from the wheel end of the lever arm, is multiplied by the *square* of the lever ratio. This general result applies to all transduction elements.

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Table 2 Ideal Linear Lumped Two-Ports

Domain to Domain	Name	Figure Number	Governing Equations (Left to Right in Figures)
Translation–translation	Lever, L = distance to function	3*a*	$F_1 = \frac{L_2}{L_1} F_2$
			$v_1 = \frac{L_1}{L_2} v_2$
Rotation–rotation	Gears, N = number of teeth	3*b*	$\tau_1 = \frac{N_1}{N_2} \tau_2$
			$\omega_1 = \frac{N_2}{N_1} \omega_2$
Rotation–translation	Crank ($\theta \ll 1$) or rack and pinion	3*c* 3*d*	$\begin{cases} \tau = RF \\ \omega = \frac{1}{R} v \end{cases}$
	•	•	•
	Screw (p = length/radius)	3*e*	$\begin{cases} \tau = pF \\ \omega = \frac{1}{p} v \end{cases}$
Electrical–electrical	Transformer	3*f*	$V_1 = \frac{N_1}{N_2} V_2$
			$i_1 = \frac{N_2}{N_1} i_2$
Electrical–rotation	Permanent magnet dc motor	3*g*	$V_a = K_b \omega$
			$i_a = \frac{1}{K_m} \tau$
Electrical–translation	Voice coil	3*h*	$V_a = K_c v$
			$i_a = \frac{1}{K_c} F$
Fluid–translation	Piston area = A	3*j*	$P = \frac{1}{A} F$
			$Q = Av$
Rotation–fluid	Fixed-displacement pump-motor, displacement/radius = D	3*k*	$\tau = DP$
			$\omega = \frac{1}{D} Q$

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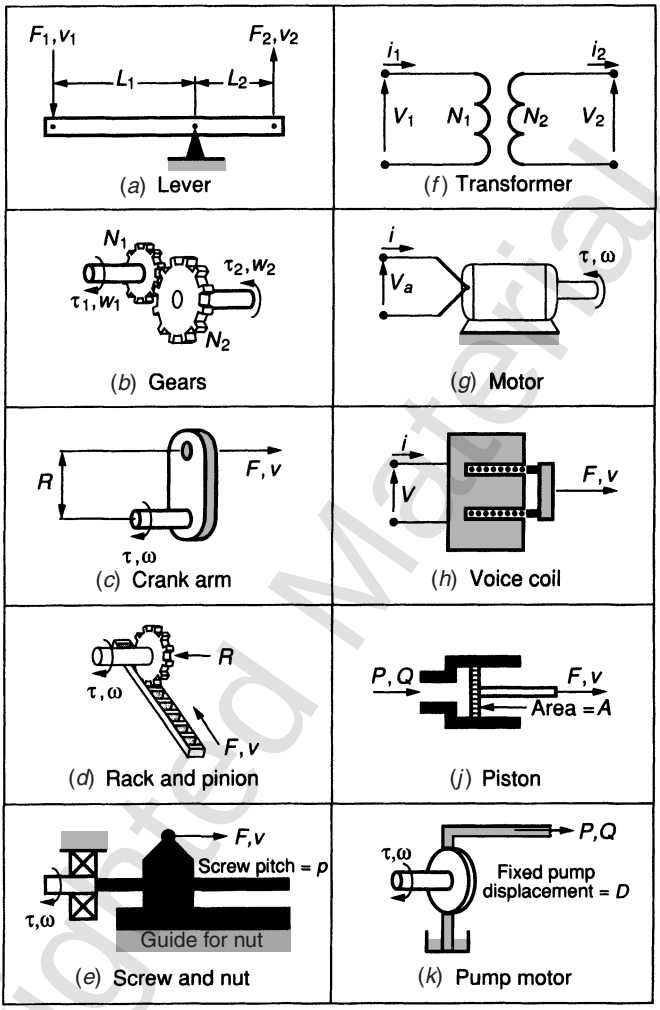


Figure 3 The ideal linear lumped two-ports: *transformers* and *gyrators.*

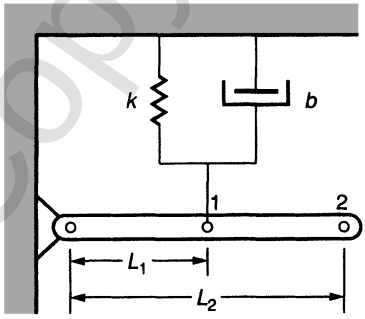


Figure 4 An abstraction of an automobile suspension linkage.

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3.6 Source Equivalents: Thévenin and Norton

Thévenin's and Norton's theorems were both originally developed for electric circuits. They are duals; that is, while Thévenin uses a series voltage source, Norton uses a parallel current source to construct an equivalent to a real subsystem being considered as a source. Both of these theorems are completely generalizable to any system, provided that the appropriate sources are used. In this chapter, the mobility analogy has been adapted so that Thévenin equivalents are constructed using sources of voltage, force, pressure, or torque, and Norton equivalents are constructed using sources of current, velocity, flow, or angular velocity. In the development below, these two classes of sources will be referred to as effort and flow sources.

Thévenin Equivalent

Assume that a subsystem being considered as a source contains within its structure one or more ideal sources. Insofar as can be measured externally at the driving point (the point of connection between any two systems considered to be source and load), any active subsystem, no matter what its load, can be replaced by a new effort source added in series with the original subsystem; all the original internal sources are set to zero. The value of the new source effort variable is the output that would appear at the driving point if the load were disconnected from the original subsystem. Setting an effort source to zero is equivalent to connecting its nodes together; setting a flow source to zero is equivalent to removing it from the system.

The output impedance of this Thévenin equivalent is clearly the impedance looking in at the driving point; it is the derivative of the driving-point effort variable with respect to the driving-point flow variable with the load disconnected. Thévenin's theorem simplifies this calculation because it tells us that the internal sources can be set to zero before the calculation is made, and the system topology is often substantially simplified by this move. We have already seen an example of this [see Eqs. (15)–(17)]. The Thévenin equivalent of this circuit is simply a new source determined with the current $i_o = 0$ (see Fig. 2):

$$V_{\text{Thévenin}} = \frac{R_3}{R_1 + R_3} V_s + \left(\frac{R_1R_2 + R_2R_3 + R_1R_3}{R_1 + R_3} \right) I_s \tag{22}$$

in series with a resistance determined with sources V_s and I_s zeroed:

$$R_{\text{Thévenin}} = \frac{R_1R_2 + R_2R_3 + R_1R_3}{R_1 + R_3} \tag{23}$$

Norton Equivalent

Assume the same subsystem considered above. Insofar as can be measured externally at the driving point, any active subsystem, no matter what its load, can be replaced by a new flow source added in parallel with the original subsystem, all the original sources set to zero. The value of the new source flow variable is the flow that would pass through a short circuit substituted at the driving point for the original load.

Referring again to Eq. (15), with V_o set to zero, we obtain the following:

$$0 = \frac{R_3}{R_1 + R_3} V_s + \left(\frac{R_1R_2 + R_2R_3 + R_1R_3}{R_1 + R_3} \right) (I_s - i_o) \tag{24}$$

we see that i_o for a short circuit would be

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$$i_o = I_s + \left(\frac{R_3}{R_1R_2 + R_2R_3 + R_1R_3}\right)V_s \tag{25}$$

and, as for the Thévenin equivalent, the circuit impedance is

$$R_{\text{Norton}} = \frac{R_1R_2 + R_2R_3 + R_1R_3}{R_1 + R_3} \tag{26}$$

Note that $R_{\text{Thévenin}} = R_{\text{Norton}}$, *always.*

4 OPERATING POINT OF STATIC SYSTEMS

A static system is a system without energy storage, a system in which there are only sources, transducers, and dissipation elements. Such systems have no transient response: They respond instantly to their inputs algebraically. The relationships among any of their variables are proportionalities—simple static gains. If the inputs to a stable dynamic system are held constant for long enough, it will become stationary; its variables will not change with time provided only that there is sufficient dissipation in the system to damp out any oscillations. Such a system is not static; it is at steady state. There is no exchange of energy among its energy storage elements, and the dissipative elements completely determine the state of the system.

4.1 Exchange of Real Power

If one system is supplying real power to another system in steady-state operation, then for the purposes of a static analysis, energy storage elements can be ignored. Both the source and the load can be considered to be purely resistive. If the source is separated from the load at the point of interest (at least conceptually), then the characteristics of source and load can be measured or computed. The load will be a power absorber—an electrical fluid resistance or a mechanical damper—and its characteristics can be represented as a line in a power plane coordinate system: voltage versus current, force versus velocity, torque versus angular velocity, or pressure versus flow. There is no requirement that this line be straight, and except that the measurement might be more difficult, there is no necessity that this line be single valued or that it start at the origin. Figure 5 shows a selection of common dissipative load characteristics.

Similarly, when the source portion of the system is loaded with a variable dissipation, the line representing its output characteristics can be plotted on the same coordinates as the load. Such characteristics are often given for pumps, servomotors, transistors, hydraulic valves, electrical supplies, and fans. Again, there is no requirement that this line be straight or that it be single valued, but it is very unusual for it to pass through the origin. If the source or the load is not constant in time or can be controlled, then a family of these characteristics will be required for variations in the parameters of the source or load. Figure 6 shows a selection of these sources. With very few exceptions, real sources cannot operate at the maximum values of their power variables simultaneously: A battery cannot deliver its maximum voltage and current at the same time. In more general terms, this means that in spite of local variations, real source characteristics tend to droop from left to right in the power plane; their average slope is negative.

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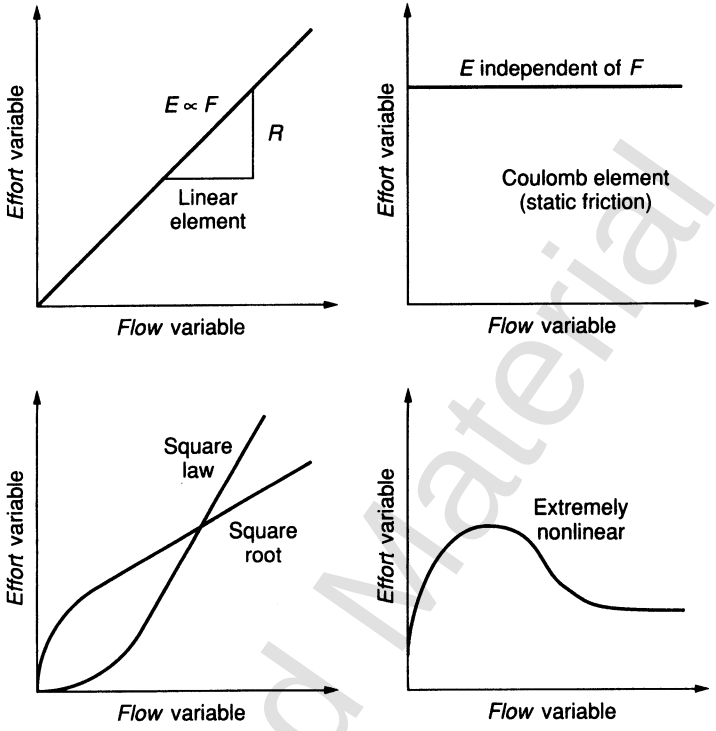


Figure 5 Common dissipative load characteristics.

4.2 Operating Points in an Exchange of Power or Energy

When resistive source and load characteristics are plotted on the same coordinates, they intersect at least once. The coordinate values at that point, or at those points if there are several, are the values of the power variables at which that combination of source and load must operate if they are connected. This is called an operating point. From a computational point of view, the source *causes* one of the power variables given the other and the load causes the other given the source. They must operate at the same point in the power plane to satisfy continuity (common flows) and compatibility (common efforts) conditions.

When there are multiple intersections, all are possible operating points, but not all will be stable operating points; for example, any disturbance from equilibrium might result in a transition to another operating point. The condition for a stable intersection is best seen graphically in Fig. 7. For a stable intersection, as shown on the left, it is required that a small perturbation of the load, which increases its demand for power, be countered by a shortage of power from the supply side of the system and a small perturbation of the load, which decreases its demand for power, be met with an excess from the source. In either case, the load will be driven back to the intersection by the excess or deficit in the source capability. A reversal of these conditions is an unstable operating point because disturbances will be driven further in the direction of their initial departure.

At the unstable intersection in Fig. 7 (2, on the right-hand side), a slight increase in the flow demand of the load will result in an overwhelming increase in the supply flow available to drive the load, which will then cause a traversal to point 1 in the figure. Similarly, if the

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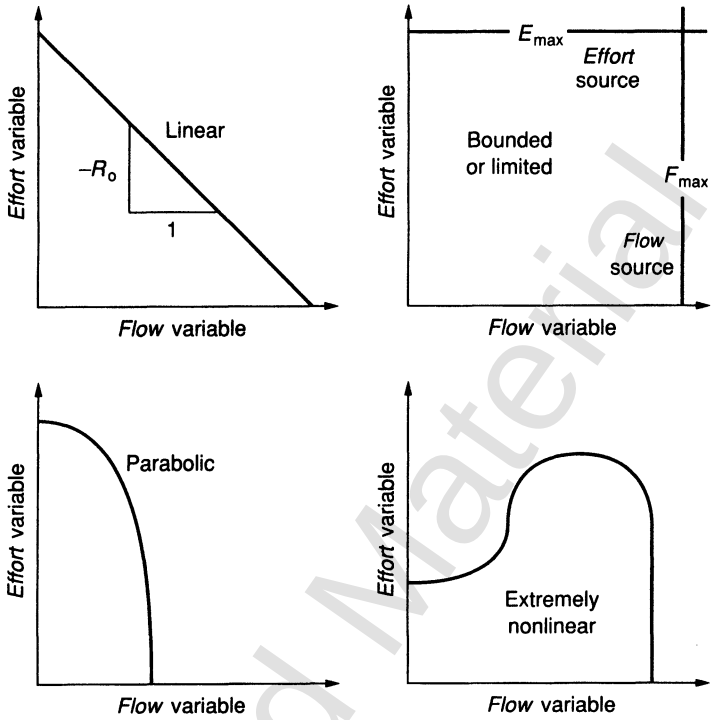


Figure 6 Common dissipative load characteristics.

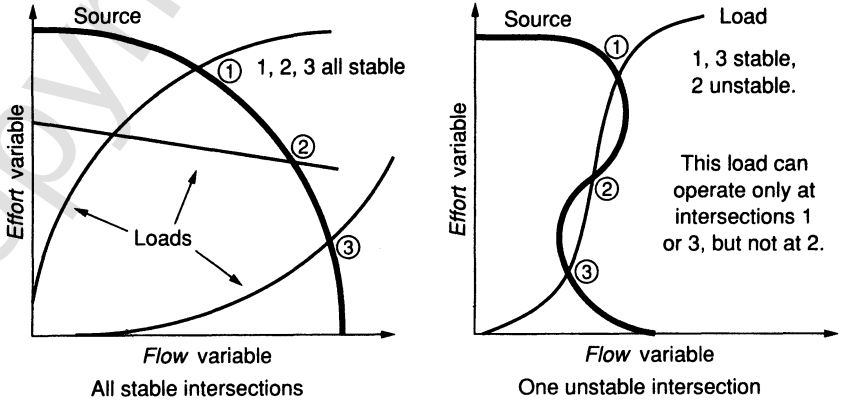


Figure 7 Stable and unstable operating point intersections.

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effort decreases slightly from operating point 2, then the source will be starved compared to the demand of the load at that effort level, so the effort will fall until point 3 is reached.

4.3 Input and Output Impedance at the Operating Point

Lines of constant power are hyperbolas in the power plane, with increasing values of the power at increasing distance from the origin. The usual sign conventions imply that sources deliver power in the first and third quadrants while loads absorb power in those quadrants. Conversely, sources absorb power in the second and fourth quadrants and loads return it. The output impedance of a source is defined as minus the slope of the output characteristic. For nonlinear characteristics, the output impedance at any point is defined as minus the slope at that point. For loads, the input impedance is defined as the slope of the load line, but for nonlinear characteristics there are two possibilities of importance: the slope of the line at a point (the incremental or local input impedance) and the slope of the chord to the point from the origin (the chordal impedance). Figure 8 summarizes these features of the power plane.

4.4 Operating Point and Load for Maximum Transfer of Power

Consider a battery with the voltage–current characteristic shown in Fig. 9. The maximum unloaded terminal voltage (open circuit) of the battery is V_{oc} volts and the short-circuit current is i_{sc} amperes. The equation of the line shown is

$$V_t = V_{oc} - R_o i_b$$

or

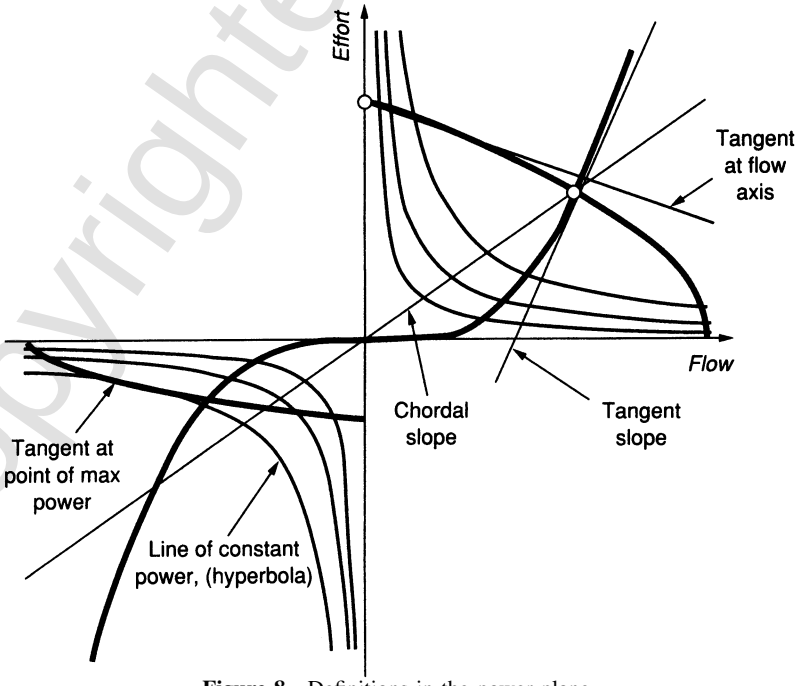


Figure 8 Definitions in the power plane.

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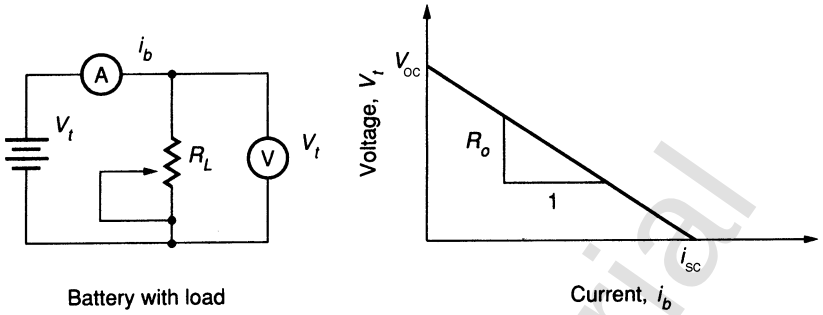


Figure 9 A battery with load.

$$i_b = i_{sc} - \frac{V_t}{R_o} \tag{27}$$

where V_t is the terminal voltage and i_b is the battery current.

The output impedance $Z_o = -dV_t/di_b = R_o = V_{oc}/i_{sc}$ (a pure resistance in this case). If the battery is loaded by a resistor across its terminals (R_L), the terminal voltage must be

$$V_t = R_L i_b \tag{28}$$

Solving Eqs. (27) and (28) simultaneously for V_t and i_b yields the following operating point coordinates:

$$V_t = \frac{V_{oc}}{1 + R_o/R_L} \qquad i_b = \frac{V_{oc}}{R_o + R_L} \tag{29}$$

The output power at the operating point is [from (29)]

$$\mathcal{P} = V_t i_b = \frac{(V_{oc})^2 R_L}{(R_o + R_L)^2} \tag{30}$$

Clearly, the output power depends on the load resistance. If R_L is zero or infinite, no power is drawn from the battery. To measure V_{oc}, we would want a voltmeter with an infinite input impedance, and to measure i_{sc}, we would want an ammeter with zero resistance. In practice, we would use a voltmeter with an input resistance very large compared to R_o and an ammeter with a resistance very small compared to R_o.

If our objective is to deliver power, then a best value of R_L is that for which the derivative $d\mathcal{P}/dR_L = 0$. This value is the point at which $R_L = R_o$. Alternatively, the maximum power output of the battery for any load occurs at the current (i_b) that maximizes $V_t i_b$. Equation (27) can be restated as

$$V_t = V_{oc}\left(1 - \frac{i_b}{i_{sc}}\right) \tag{31}$$

so that

$$\mathcal{P} = V_t i_b = V_{oc}\left(1 - \frac{i_b}{i_{sc}}\right) i_b \tag{32}$$

which is maximized at $\mathcal{P} = V_{oc} i_{sc}/4$ when $i_b = i_{sc}/2$.

For the battery characteristic, the operating point $i_b = i_{sc}/2$ yields $V_t = V_{oc}/2$ [substitution in Eq. (29)]. A loading resistor characteristic must pass through this point to draw

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maximum power from the battery, so that the load resistance must be $R_L = V_{oc}/i_{sc} = R_o$. At this operating point, the equivalent resistor within the battery (representing the internal losses in the battery) is dissipating exactly as much power as is being delivered to the load.

If we want maximum power delivery, impedance should match the load to the source, but if we want to minimize power delivery from a source, then impedance mismatching is the key. Impedance matching assures that the source and load will divide the power equally; all other impedances will result in less power transfer.

4.5 An Unstable Energy Exchange: Tension-Testing Machine

Although tensile studies of material properties require only a simple test apparatus, it is not simple to interpret the data from such tests. The problem is that the tensile test machine and the specimen can interact[3] in an unstable way. Almost any desired stress–strain curve can be obtained in a given material by a suitable choice of the test machine's elastic compliance compared to the specimen.

A tensile test involves the interaction between two springs, one that represents the specimen and the other combined with a velocity source that represents the testing machine. Figure 10 shows this simple model. While the testing machine is linearly elastic and does not yield, the test specimen is not elastic. It undergoes a large plastic deformation in a typical load–elongation test. Normally in such a test, the specimen is to be elongated at a constant cross-head velocity (v). The test machine, however, is not a velocity source as is commonly supposed; that source is really in series with a spring (K) representing the elastic deformations of the testing machine structure between the source of the motion and the jaws of the machine.

In the course of a test, the specimen undergoes an elongation (δ) comprised of both elastic and plastic displacements. The test machine undergoes only an elastic displacement (ξ) given by F/K, where F is the load applied to the specimen by the machine at a given cross-head displacement (y), which is really the sum of both the specimen (δ) and the machine (ξ) displacements. Figure 11 shows the components of this force–displacement situation.

The cross-head velocity of the test machine is made up of two components as well:

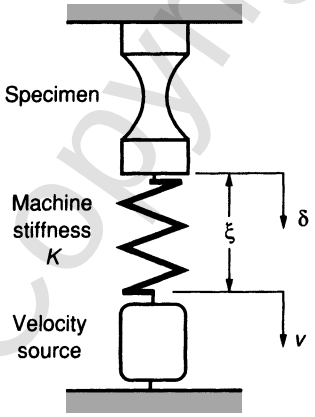


Figure 10 Simple model of a tensile-testing machine.

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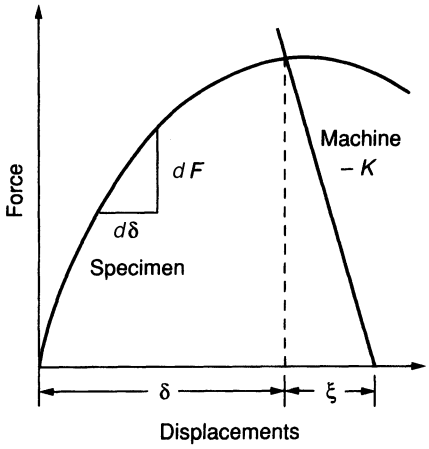


Figure 11 Components of the force–displacement curve.

$$v = \frac{dy}{dt} = \frac{d\delta}{dt} + \frac{d\xi}{dt} \tag{33}$$

The machine displacement (ξ), however, is a function of the force being applied, so it is therefore a function of the elongation of the specimen (δ). Equation (33) must be rewritten:

$$v = \frac{d\delta}{dt} + \frac{d\xi}{d\delta}\frac{d\delta}{dt} = \frac{d\delta}{dt}\left(1 + \frac{d\xi}{d\delta}\right) \tag{34}$$

Since $\xi = F/K$, presuming linearity for the machine, we get the following derivative:

$$\frac{d\xi}{d\delta} = \frac{1}{K}\frac{dF}{d\delta} \tag{35}$$

where the term $dF/d\delta$ is the slope of the force–elongation curve of the specimen. The slope, in other words, is the driving-point stiffness at any point along the test curve. If these last two equations are combined, the specimen's elongation rate is found in terms of the cross-head velocity (v), which is normally constant, the machine stiffness (K), and the driving-point stiffness of the machine ($dF/d\delta$):

$$\frac{d\delta}{dt} = \frac{v}{1 + \dfrac{1}{K}\dfrac{dF}{d\delta}} \tag{36}$$

As a test proceeds, however, the specimen starts to neck down, its stiffness begins to decrease, and then it becomes negative. When the driving-point stiffness ($dF/d\delta$) passes through zero, the specimen has reached the maximum force. The stiffness thereafter decreases until it equals $-K$ and at that point the specimen elongation rate ($d\delta/dt$) becomes infinite for any cross-head rate (v), and the system is mechanically unstable. This instability point in a tensile test is the point at which the machine load line is tangential to the load–elongation curve of the material being tested. At that point, the specimen breaks and the energy stored in the machine structure dumps into the specimen at this unstable intersection.

Clearly, percent elongation at failure specifications is not very meaningful because it depends on the test machine stiffness. Figure 12 shows that for accurate measurements the

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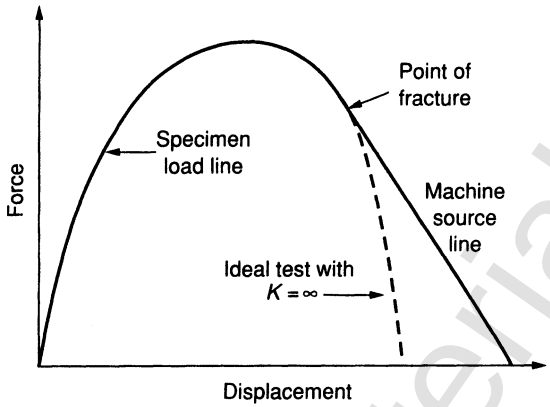


Figure 12 The point of instability in a tensile test.

stiffness of the test machine must be many times as large as the stiffness of the specimen near failure or the machine will dump its energy into the specimen and force a premature failure.

4.6 Fatigue in Bolted Assemblies

The mechanical engineering reader will perhaps recall that preloading a bolt stretches the bolt and compresses the part being bolted, but it is the relative stiffness of the bolt and part that determines what fraction of external loads applied to separate the part from the bolt will be felt by the bolt.* If the objective is to relieve the bolt of these loads, as it is in the head bolts of an automobile engine, then the designer tries to make the part much stiffer than the bolt; he or she mismatches the stiffness of the bolt and the part by specifying a hard head gasket. If the stiffnesses of the bolt and the part were the same, then they would share the external load equally: The bolt tension would increase by half the applied load while the part compression would decrease by half the applied load. If the gasket were very soft compared to the bolts, then the bolts would see virtually all of the applied load.

4.7 Operating Point for Nonlinear Characteristics

Let us continue to use the battery as an example of a linear source. It should be obvious that the maximum power point for the battery is independent of the load it must drive, but the load characteristic, however nonlinear, must pass through this point for maximum power transfer. Figure 13 illustrates this. It is not the slope of the load impedance that must match the source impedance, it is the *chordal slope,* the slope of a line from the origin of the power plane to the maximum power point of the source.

An orifice supplied from a constant upstream pressure and loaded at its output is a good example of a nonlinear source. For a short, sharp-edged orifice, the orifice equation† applies:

---

*Refer to any text on machine design under the indexed heading "fatigue in bolts."

†This is derived from Bernoulli's equation. The discharge coefficient (C_d) corrects for viscous effects not considered and for the existence of a vena contracta or convergence in the flow through the orifice, which makes the area of the jet smaller than the orifice itself. For most oils $C_d \approx 0.62$.

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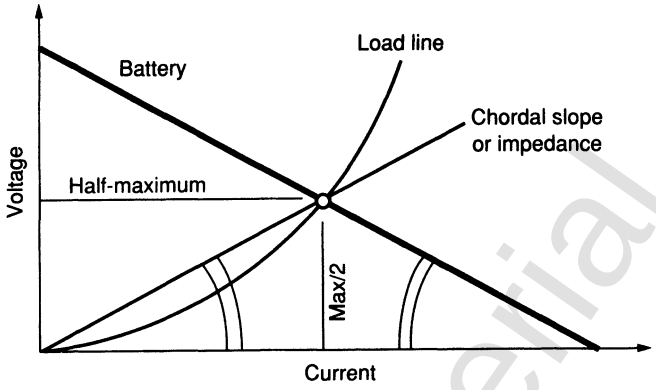


Figure 13 Chordal impedance matching.

$$Q_o = C_d A_o \sqrt{\frac{2}{\rho}\Delta P} = C_d A_o \sqrt{\frac{2}{\rho}(P_{\text{up}} - P_{\text{down}})} \tag{37}$$

where Q_o is the orifice flow, C_d is the discharge coefficient of the orifice, A_o is the orifice area, ρ is the density of the fluid, and $\Delta P = P_{\text{up}} - P_{\text{down}}$ is the pressure drop across the orifice. If the upstream pressure is kept constant, then P_{down} may be considered the output pressure and Q_o may be considered the output flow from this orifice, a characteristic typical of many hydraulic valves. Where is the maximum power point on this characteristic?

Hydraulic power $\mathcal{P}$ is the product $Q \cdot P$. It is a maximum when it is stationary with respect to either Q or P. If we nondimensionalize Eq. (37) with respect to the maximum flow through the orifice when $P_{\text{down}} = 0$, then the orifice equation becomes the following:*

$$\mathbf{Q}^* = \frac{Q_o}{Q_{\max}} = \sqrt{1 - \frac{P_{\text{down}}}{P_{\text{up}}}} = \sqrt{1 - \mathbf{P}^*} \tag{38}$$

$$\mathcal{P}^* = \frac{\mathcal{P}}{\mathcal{P}_{\max}} = \mathbf{P}^*\mathbf{Q}^* = \mathbf{P}^*\sqrt{1 - \mathbf{P}^*} \tag{39}$$

$$\frac{d\mathcal{P}^*}{d\mathbf{P}^*} = \sqrt{1 - \mathbf{P}^*} - \frac{\mathbf{P}^*}{2\sqrt{1 - \mathbf{P}^*}} = 0 \tag{40}$$

for which $\mathbf{P}^* = \frac{2}{3}$, and by substitution into (38), the flow at that point is $\mathbf{Q}^* = 1/\sqrt{3}$, and the maximum power delivered is $(\sqrt{2}/3)P_{\max}Q_{\max}$. The output admittance of the orifice is the slope of the curve at any operating point:

$$Z_o = \frac{d\mathbf{Q}^*}{d\mathbf{P}^*} = \frac{1}{2\sqrt{1 - \mathbf{P}^*}} = \left.\frac{\sqrt{3}}{2}\right|_{\mathbf{P}^*=2/3} \tag{41}$$

The load for which maximum power will be delivered, whether it has a linear or nonlinear characteristic, must pass through the maximum power point. Its *chordal admittance* must be

*Where bold letters will be used to indicate the nondimensional forms.

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$$Z_{\text{chordal}} = \frac{\mathbf{Q}^*_{\text{op}}}{\mathbf{P}^*_{\text{op}}} = \frac{1/\sqrt{3}}{2/3} = \frac{\sqrt{3}}{2} \tag{42}$$

which is exactly the slope in Eq. (41). Figure 14 shows this graphically.

Note that for this power plane, the ordinate is *flow* and abscissa is *pressure* so that the slopes shown are admittances rather than impedances. While in other figures the abscissa has always been effort and the ordinate flow, it is conventional in hydraulic systems to show these figures the other way probably because pressure is usually the independent variable in hydraulic system characteristics: Pressure is controlled and flow is measured.

4.8 Graphical Determination of Output Impedance for Nonlinear Systems

A three-way hydraulic valve supplies or drains its load through a pair of variable orifices, one connecting the supply to the load and the other connecting the drain or tank to the load. These orifices are operated (typically on a single moving part of the valve) in a push–pull fashion; that is, as one opens, the other closes. If both orifices are partially open together when the valve is centered, the three-way valve is open centered. If one is wide open just as the other closes, the valve has no overlap.

Suppose it is required to find the P–Q characteristics of the load port for flows both into and out of the valve. The system is shown in Fig. 15. Thus for the upstream and downstream orifices respectively, Eq. (37) becomes

$$Q_u = C_d A_u \sqrt{\frac{2}{\rho}(P_S - P_L)} \quad \text{and} \quad Q_d = C_d A_d \sqrt{\frac{2}{\rho}(P_L - P_T)} \tag{43}$$

The load flow (Q_L) is the difference between Q_u, and Q_d. Also, the tank pressure (P_T) can be taken as zero. These equations combined become

$$Q_L = C_d A_u \sqrt{\frac{2}{\rho}(P_S - P_L)} - C_d A_d \sqrt{\frac{2}{\rho} P_L} \tag{44}$$

It is much more convenient to work with Eq. (44) in dimensionless form. If we assume that the maximum upstream and downstream areas are the same and that they are truly push–

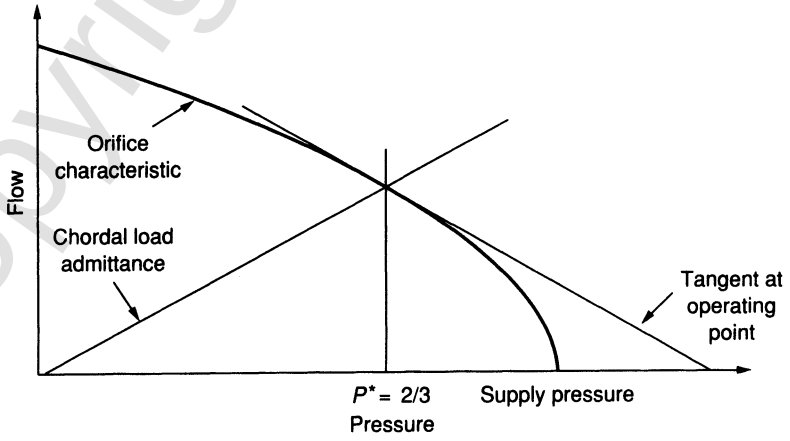


Figure 14 Nonlinear impedance matching.

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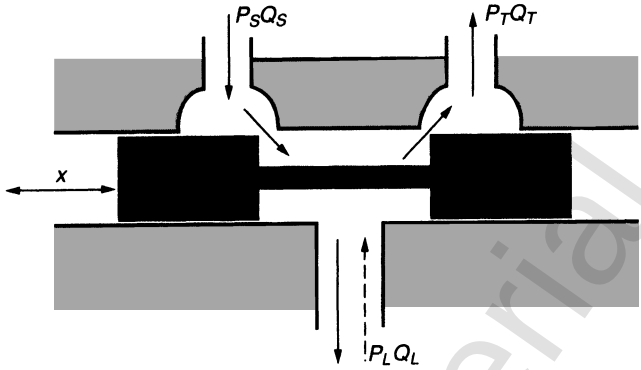


Figure 15 A three-way valve geometry and output.

pull, then we can nondimensionalize the area using $A_{\max}$. The discharge coefficient (C_d) is a constant for conventional valve geometries. The supply pressure (P_S) is a convenient term for the pressure nondimensionalization. Flows can be nondimensionalized with respect to a maximum flow that would pass through either orifice at full area $A_{\max}$ with P_S acting across it:

$$Q_{\max} = C_d A_{\max} \sqrt{\frac{2}{\rho} P_S} \tag{45}$$

If we set $P_L/P_S = \mathbf{P}$ and $Q_L/Q_S = \mathbf{Q}$ and express the upstream and downstream orifice sizes as push–pull fractions of $A_{\max}$, $0.5(1 + x)A_{\max}$ upstream and $0.5(1 - x)A_{\max}$ downstream, where $-1 \le x \le 1$ and the valve is centered for $x = 0$, then Eq. (44) combined with (45) yields

$$\mathbf{Q} = \mathbf{Q}_u - \mathbf{Q}_d = 0.5(1 + x)\sqrt{1 - \mathbf{P}} - 0.5(1 - x)\sqrt{\mathbf{P}} \tag{46}$$

which is one of those unfortunate equations in which the radical cannot be eliminated by squaring both sides. While Eq. (46) can be readily plotted, $\mathbf{Q}$ versus P with x (the valve stroke) as a parameter, it is instructive to construct it instead from its parts.

The term $0.5(1 + x)\sqrt{1 - \mathbf{P}}$ is the characteristic family for the upstream orifice, and the term $0.5(1 - x)\sqrt{\mathbf{P}}$ is the characteristic family for the downstream orifice. The first of these are parabolas to the left (on their sides because they are roots), starting at $\mathbf{Q}_u = 0.5(1 + x)$, $\mathbf{P} = 0$ and ending at $\mathbf{Q}_u = 0$, $\mathbf{P} = 1$ (there is no flow when the downstream pressure equals the upstream pressure). The second term starts at $\mathbf{Q}_d = 0$, $\mathbf{P} = 0$ and rises to the right to the points $\mathbf{Q}_d = 0.5(1 - x)$, $\mathbf{P} = 1$. All this is shown in Fig. 16. If there is no load *flow,* then the curves for the upstream orifice show its output characteristic while those for the downstream orifice represent the only load. The intersections predict the operating pressures for $\mathbf{Q} = 0$ as the valve is stroked, $-1 \le x \le 1$.

If there is a load flow ($\mathbf{Q}$), then continuity must be served. This requires that

$$\mathbf{Q} = \mathbf{Q}_u - \mathbf{Q}_d \rightarrow \mathbf{Q}_u = \mathbf{Q} + \mathbf{Q}_d \tag{47}$$

In Fig. 16, $\mathbf{Q}$ is simply a vertical bar between the curves for $\mathbf{Q}_u$ and $\mathbf{Q}_d$ whose length is the load flow. Thus for any load pressure value on the abscissa, the load flow is determined as the vertical distance between the input orifice curve and the output orifice curve. The load flow is positive if the upstream orifice curve is above the downstream orifice curve but

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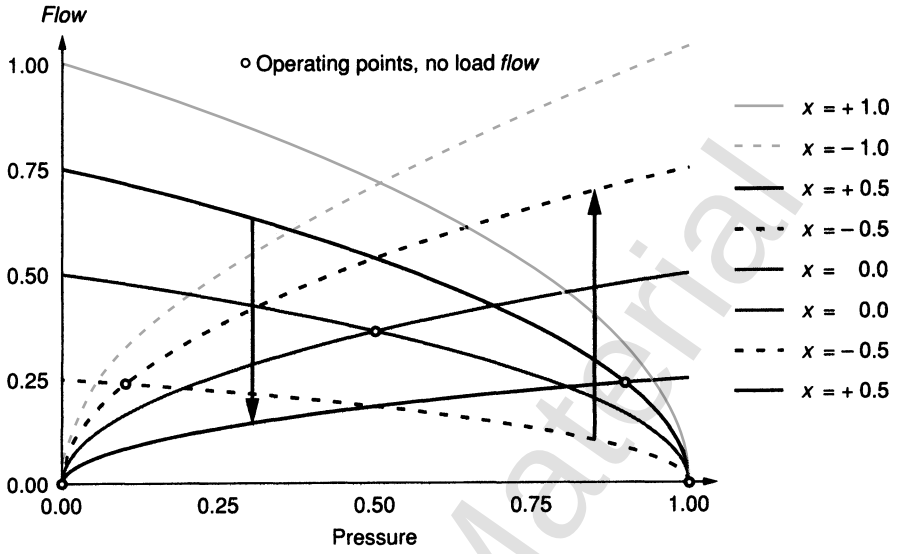


Figure 16 Flow versus pressure drop characteristics for the valve orifices.

negative otherwise. These data can be picked off and plotted as the **Q–P** characteristic of the valve, as shown in Fig. 17.

Figure 17 illustrates that an operating point on an input–output characteristic forces compatibility and continuity, but as long as those are preserved, a second energy exchange as at the $P_L - Q_L$ port of this valve can still be accommodated. This approach can be extremely useful in systems such as air-conditioning ducting where the fan characteristic is known only graphically, and then its output impedance at some point along the ducting must be determined. Exactly the same procedure is used.

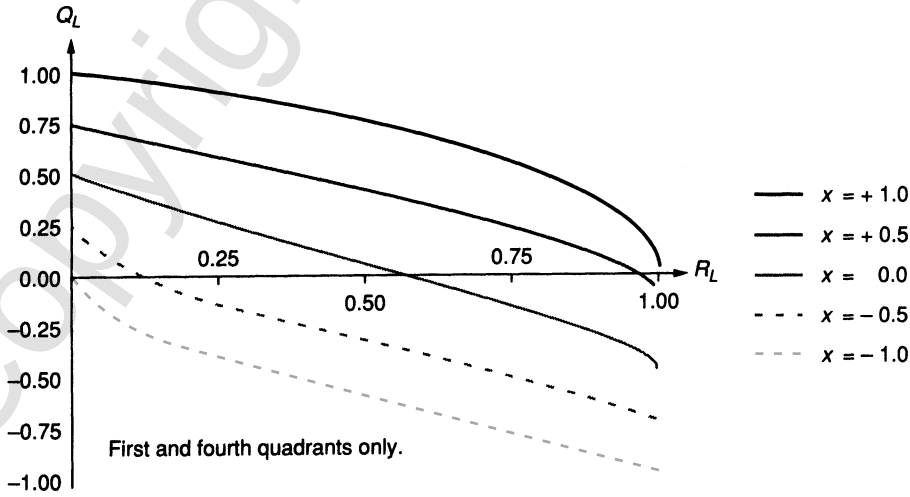


Figure 17 The output characteristic of the valve.

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5 TRANSFORMING THE OPERATING POINT

It is often not among the system designer's options to choose either the output impedance of the power source in the system or the input impedance of the load that must drive. The only recourse at that point is to insert a transformer, transducer, or gyrator in the system if it does not already contain one or to vary the modulus of the two-port element if it does. In the old tube-type audio systems, the output impedance of the push–pull power tubes exceeded 1000 Ω while the input impedance of the speakers available then was 4, 8, or 16 Ω. To match the amplifier to the load, each channel had a large transformer for speaker connections with taps having turns ratios of $\sqrt{1000/4}$, $\sqrt{1000/8}$, and $\sqrt{1000/16}$. With this arrangement, maximum power transfer was assured down to the lowest frequencies for which the transformers were designed.

5.1 Transducer-Matched Impedances

Suppose a permanent direct current (dc) magnet servomotor is to drive a screw that, in turn, drives a mass, perhaps a machine-tool table. If the objective is to minimize the move time from stationary start to full stop, then the optimal trajectory for the servo, assuming equal and constant acceleration and deceleration, is well known: maximum acceleration to either half of the move distance or maximum velocity, whichever comes first, followed by maximum deceleration to the finish. These trajectories are illustrated in Fig. 18.

The motor operates in two modes: at maximum acceleration (maximum torque), which is set by the maximum short-term current permitted by the coercivity of the motor magnets and the capacity of the commutation, and at maximum speed, if that is reached, set by the maximum voltage available or by whatever the commutation allows. Often, servo designers accomplish these two modes by using an overvoltage (two to three times rating) during acceleration and deceleration, which runs the power amplifier as a current source, and then as maximum speed is attained, switching the amplifier voltage limit to the motor rating, which runs the amplifier as a voltage source. This achieves a constant acceleration and a constant top speed.*

During the acceleration or deceleration phase of the trajectory, the electrical torque available is accelerating two inertias: the motor itself and the load. Since the motor is acting

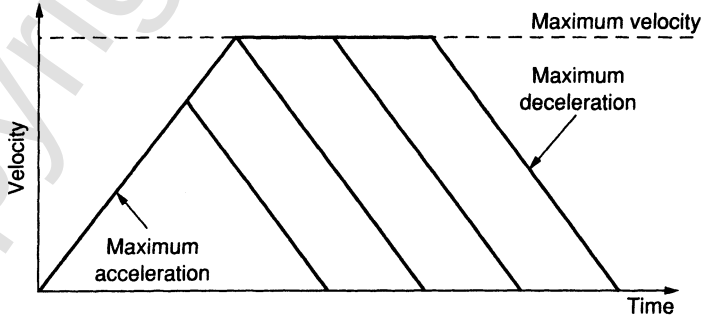


Figure 18 Optimal trajectories for point-to-point move.

*Actually, designers rarely use maximum acceleration or maximum velocity because such full-scale values leave no overhead for control. A servo running in saturation is an open-loop system.

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as an electromagnetic torque source while constant current is supplied, $\tau_{\text{motor}} = K_m i_{\text{armature}}$, the mechanical output impedance viewed at the motor shaft is the rotor inertia between the output shaft and the electromagnetic torque source:

$$\boldsymbol{Z}_{o|\substack{\text{motor shaft with}\\ \text{constant armature current}}} = \boldsymbol{J}_{\text{armature}}\boldsymbol{s} \tag{48}$$

During acceleration and deceleration, the optimal load impedance (Z_{load}) will be equal to the motor inertia, and the available electromagnetic torque will be shared equally by the motor armature and the load.

The load given in this example is primarily massive, so with reflection through the screw, the load inertia is computed as the load inertia times the square of the transducer ratio (p for the screw; see Table 2):

$$Z_{\text{load}} = p^2 M_{\text{load}} s \tag{49}$$

To the extent possible, the pitch of the screw or the inertia of the motor should be chosen to achieve this match. Failing both of those options, a gear box should be placed between the screw and the motor to accomplish the match required.

5.2 Impedance Requirements for Mixed Systems

When a source characteristic is primarily real or static (so that the source impedance is resistive) and the load is reactive or dynamic (dominated by energy storage elements), then impedance matching in the strictest sense is impossible, and the concept of passing the load line through the source characteristic at the maximum power point does not make sense. How then does one match a static source to a dynamic load or the reverse?

Figure 19 shows an electrohydraulic position servo driving a mass load with negligible damping losses, and Fig. 20 shows the pressure flow characteristics of the servovalve: a family of parabolas in the power plane used for hydraulic systems. The transducer between this hydraulic power plane and the force–velocity power plane in which the load operates is a piston of area A. There are two equations:

$$P = \frac{1}{A} F \qquad \text{and} \qquad Q = Av \tag{50}$$

With these equations, a load in F–v coordinates can be transformed to a load in P–Q coordinates for superposition on the source characteristics.

The impedance of the mass load, however, is simply $Z_{\text{load}} = (M)(s)$, but this cannot be plotted in an F–v coordinate system because it is the slope of a straight line in energetic

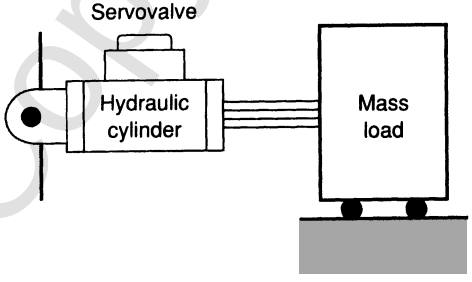


Figure 19 Electrohydraulic position servo.

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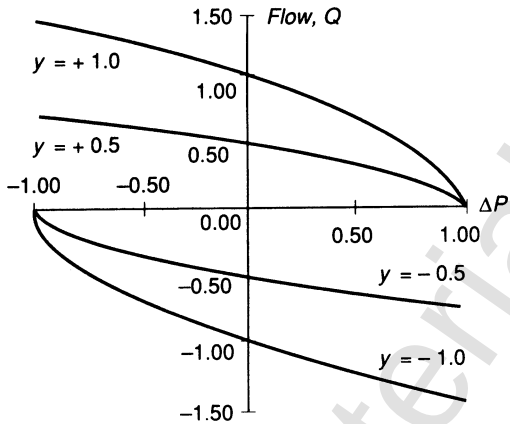


Figure 20 Servovalve characteristic.

coordinates: F versus dv/dt. To match load to source, we need the force–velocity relationship for the load. A servo designer normally has a frequency response in mind for his or her application or can determine one from a linearization of the system. Suppose, in this example, that the valve operating between a supply pressure (P_s) and a maximum load flow ($Q_{\max}$) must drive the mass load (M) through the piston of the area (A), with a frequency response flat in position to a frequency of ω (radians per second), at an amplitude of $D_{\max}$. With these conditions, the force–velocity relationship for the load can be found as follows.

If y is the displacement, $\dot{y}$ is the velocity, and $\ddot{y}$ is the acceleration, then the most taxing demand on the servo will be $y = D_{\max} \sin(\omega t)$, where t is time. Then $\dot{y} = D_{\max}\omega \cos(\omega t)$ and $\ddot{y} = -D_{\max}\omega^2 \sin(\omega t)$. Given that $F = M\ddot{y}$ is the load equation, however, the load can now be expressed parametrically as a pair of equations:

$$v = \dot{y} = D_{\max}\omega \cos(\omega t) \qquad \text{and} \qquad F = -MD_{\max}\omega^2 \sin(\omega t) \tag{51}$$

If these are cross-plotted in the force–velocity plane with time as a parameter, the plot traces out an ellipse as time goes from zero through multiples of $2\pi/\omega$. Transforming these to the P–Q plane requires application of the piston equations (50) to yield

$$Q = AD_{\max}\omega \cos(\omega t) \qquad \text{and} \qquad P = -\frac{MD_{\max}\omega^2}{A} \sin(\omega t) \tag{52}$$

If the valve is to drive the mass through the piston around the trajectory, $y = D_{\max} \sin(\omega t)$, then the valve output characteristic for the maximum valve stroke must *entirely enclose* the elliptical load characteristic derived in Eqs. 52. Furthermore, if the valve and load are to be perfectly matched, then the valve characteristic and the load ellipse must be *tangent* at the maximum power point for the valve: $P = \frac{2}{3}P_S$, $Q = (1/\sqrt{3})Q_{\max}$. This is shown in Fig. 21.

Note that this requirement means that neither maximum valve output pressure nor maximum valve flow will ever be reached while the load executes its maximal sinusoid. If $P_s = -MD_{\max}\omega^2/A$ and $Q_{\max} = AD_{\max}\omega$, the valve sizing would have provided inadequate power for reaching any but those points on the trajectory, which would therefore have followed the valve characteristic instead. The correct matching relationships are those which satisfy the following equations:

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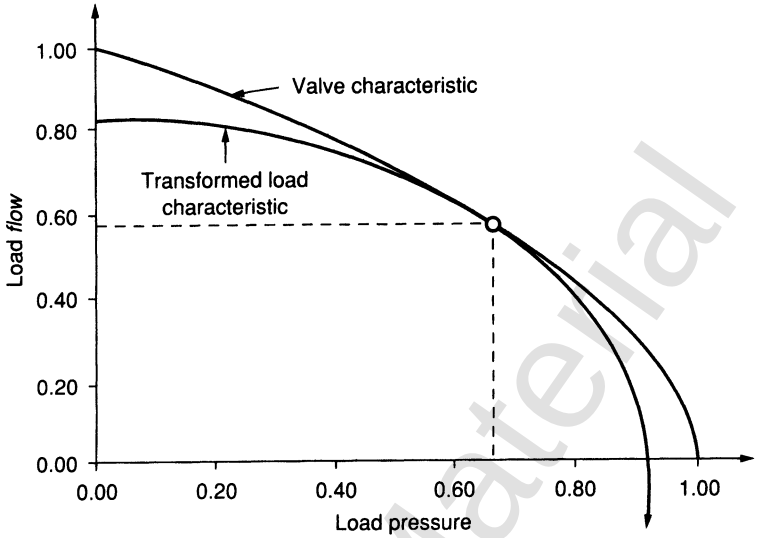


Figure 21 Matching power requirements for a dynamic load.

$$\frac{1}{\sqrt{3}} Q_{\max} = AD_{\max}\omega \cos(\omega t) \quad \text{and} \quad \frac{2}{3} P_s = \frac{MD_{\max}\omega^2}{A} \sin(\omega t) \tag{53}$$

at the appropriate time. These equations relate $D_{\max}$, ω, M, $Q_{\max}$, P_s, and A, so any five of these determine t and the sixth. If the load mass, supply pressure, piston area, peak frequency, and maximum amplitude are all known, for example, Eqs. (53) size the valve by determining its maximum required flow. If the valve has been selected, these equations will size the piston.

In the example above, the load was purely massive. Most real loads are dissipative as well. The procedure outlined above, however, need only be modified by adding the damping term to the force equation:

$$F = M\ddot{y} + B\dot{y} = -MD_{\max}\omega^2 \sin(\omega t) + BD_{\max}\omega \cos(\omega t) \tag{54}$$

where B is the damping coefficient. This has the effect of tipping the axis of the elliptical load line up to the right, but the remainder of the development follows as before with the added term. Figure 22 shows the result. If the load mass were to be negligible, then the elliptical transformed load line in Fig. 22 would collapse to a line passing through the maximum power point of the valve characteristic, as discussed previously.

6 MEASUREMENT SYSTEMS

Measurement extracts information about the state of a measured system, usually in the form of one of the factors in the measurement domain whose product is power or energy flux, such as voltage or current, pressure or flow, and so on. A measurement usually extracts some energy from the system being measured, if only in a transient sense; each link in the chain from measurement to display involves a further exchange of energy or power. If we measure

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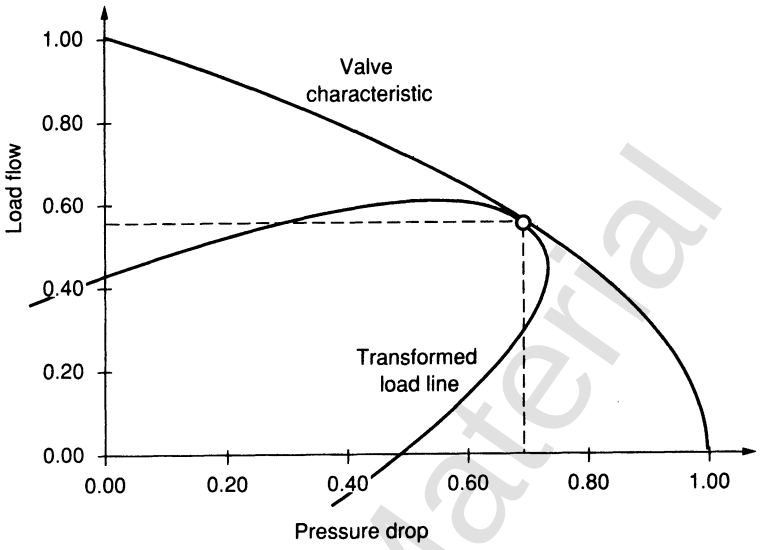


Figure 22 Matching power requirements for a mass load with dissipation.

voltage in a circuit, for example, we must draw some current, at least instantaneously, so that the power is dissipated by effective resistances, or energy is stored capacitively, inductively, or both.

At the measurement interface, we wish to disturb the measured system as little as possible by extracting as little energy or power as can be managed. It is usually our objective to pass power or energy along the chain of elements that form the measurement system, and there is a best combination of the energy variables to attain the optimum transfer. This chapter has dealt with these issues: the maximization of energy transfer within the system where we want it and the minimization of energy theft at the measurement interface where we do not.

6.1 Interaction in Instrument Systems

The generalized instrument consists of a number of interconnected parts with both abstract and physical embodiments. An orifice flow-metering system might consist of the following chain: An orifice plate converts the flow to a pressure drop; a diaphragm converts the pressure drop to a force; a spring (perhaps an unbonded strain-gage bridge) converts the force to a displacement; the strain gages convert the displacement to a resistance change; a bridge arrangement converts the resistance change to a differential voltage; and a galvanometer converts the voltage to a trace on paper. Figure 23 illustrates this chain.

In setting up this chain, the orifice is sized to minimize the *pressure drop* resulting from our flow measurement, and the diaphragm must be sized for minimum pumping volume during transients in pressure, or it will alter the flow reading in a transient sense. The diaphragm, however, has an output stiffness; the force it transmits decreases with increasing displacement, and it is driving the unbonded gages, converting force to displacement. The combined stiffness of the unbonded strain gages, which will be linear springs, must equal

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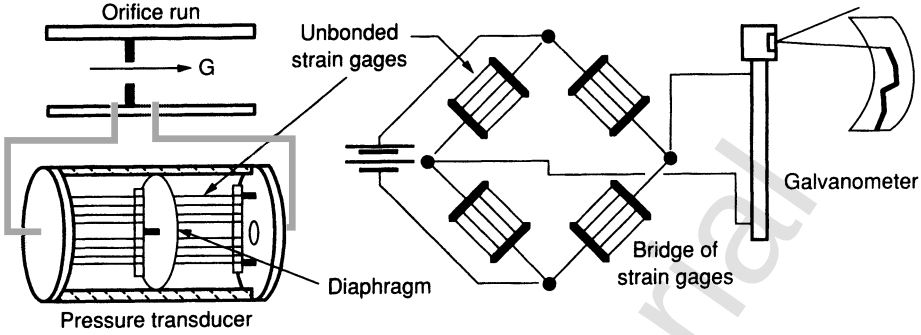


Figure 23 The chain of a flow measurement.

the average stiffness of the diaphragm* in order to ensure that the strain energy transfer is maximized for a given input energy from the fluid.

The displacement–strain relationship is a definition. Further along the chain, however, this strain is converted to a differential voltage whose magnitude depends on the input impedance of the galvanometer, which is loading the bridge. For bridges constructed with the same resistance in each arm, whether the resistors are active gages or not, the input and output impedances are the same: the resistance of one arm, usually the gage resistance. If we ignore dynamic considerations (such as galvanometer damping), the optimal galvanometer will therefore have the same resistance for maximum deflection per unit strain as one gage in the bridge.

At the measurement interface, therefore, the objective is to mismatch the impedances between the measurement system and the measured system as much as possible. There is more to this than the selections of voltage-measuring devices with higher impedance than the system in which voltage is being measured and current-measuring devices with lower impedance than the circuits in which they are placed. The measurement and control engineer must always be aware of dynamic loading as well.

The output impedance of any piezoelectric device, for example, is almost purely capacitive and is typically only a few picofarads for small devices. The input impedance of most oscilloscopes is a parallel combination of 1 MΩ and 100 pF. Either or both of those would load a piezoelectric device to near uselessness, even though for most other purposes they are *high* impedances The resistance would reduce the charge on the crystal much too quickly, and the capacitance would steal charge and reduce the voltage output drastically. An attempt to measure the pressure in a small volume with a transducer that has a large *swept* volume itself would meet with the same failure: The displacement of the transducer would alter the volume in which the pressure was measured. Holography has become popular in the study of the vibration of thin plates and shells because it does not *load* the structure by adding mass as an array of accelerometers would.

*If the diagram were rigidly supported in the center, then the force on the support would be the pressure times the effective area of the diaphragm. If the support were not rigid, this output force would depend nonlinearly on the support displacement. The plot's negative slope of force transmitted vs. displacement is the output stiffness of the diaphragm, and pressure × area is the force source in this model.

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6.2 Dynamic Interactions in Instrument Systems

It is not only the steady-state *loading* of a measurement that is of concern; under many circumstances, the unsuspected dynamics of the instrument being used will lead to erroneous results. Consider the simple measurement interface shown in Fig. 24. The readout instrument, an oscilloscope for example, has both resistance and capacitance. The load impedance is therefore the parallel combination of these:

$$Z_{\text{inst}} = \frac{R_i(1/C_i s)}{R_i + 1/C_i s} = \frac{R_i}{R_i C_i s + 1} = \frac{R_i}{\tau_i s + 1} \qquad \text{with} \qquad \tau_i = R_i C_i \tag{55}$$

Thus the load depends on the frequency of the voltage being measured. This might not concern us in the sense that we can predict it and compensate for the known phase shift which this will induce, but when the interaction of the two systems is considered, the problem becomes more obvious. The total impedance loading the measurement of V_s includes the source impedance Z_o. Suppose this is purely resistive (R_o):

$$Z_{\text{total}} = R_o + Z_{\text{inst}} = \frac{R_o R_i C_i s + (R_o + R_i)}{R_i C_i s + 1} \tag{56}$$

The readout instrument is sensitive only to the voltage it sees, which has been reduced by the voltage drop in R_o. In general terms, the measured voltage, V_{meas}, is the voltage across the instrument's input resistor and capacitor, R_i and C_i:

$$\frac{V_{\text{meas}}}{V_s} = \frac{Z_{\text{inst}}}{Z_{\text{total}}} = \frac{R_i}{R_o R_i C_i s + (R_o + R_i)} = \frac{1}{R_o C_i s + (R_o/R_i + 1)} \tag{57}$$

Unless we know the output impedance at the point of measurement, in this case R_o, we do not even know the time constant or break frequency germane to the measurement. In the event that Z_o is also complex (has reactive terms), this situation is more complicated, and if $Z_o = L_o s + R_o$, for example, the system could even be oscillatory.

Sometimes, frequency-dependent impedances are intentionally introduced into a measurement system, most commonly in the form of passive filters. Figure 25* shows a first-

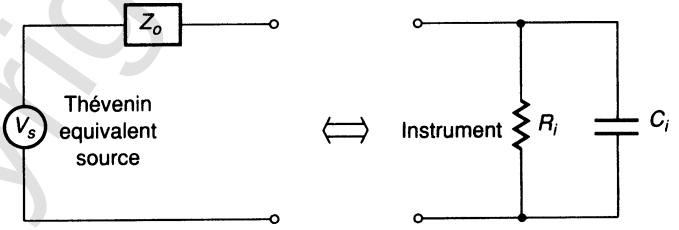


Figure 24 A measurement with a dynamic instrument.

*This example is drawn entirely from Ref. 4, with the permission of Dr. C. L. Nachtigal. In this and the previous example, Figs. 24 and 25, the source voltage is referred to as a Thévenin equivalent source. In single-loop circuits the source voltage is by definition the Thévenin voltage, since removing one element from the loop causes it to become open circuited. If the source is a sensor, for example, the magnitude of the voltage is governed by the value of the sensed variable and, of course, its own design parameters. The Thévenin, or open-circuit voltage, is specified on the sensor data sheet.

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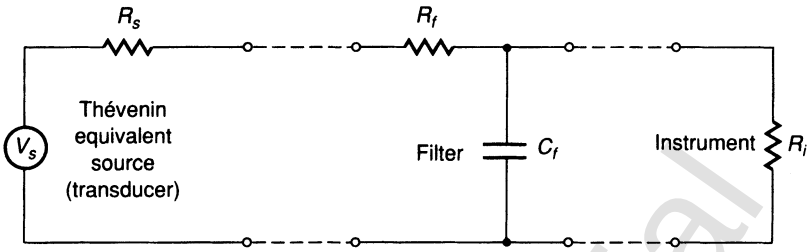


Figure 25 Schematic of cascaded instrument system.

order, low-pass filter being driven by a source with a nonzero output impedance (R_s) and being loaded by a readout instrument with finite input impedance (R_i).

The loading problem is potentially present at the interface of each stage indicated by the broken lines and terminals. What is not so obvious is that the dynamics of this filter are quite sensitive to its source and load. An unwary designer might choose the filter time constant ($\tau_f = R_f C_f$) with little regard for the values of R_f and C_f and essentially design the filter under the simple assumption that source and load were ideal.

To the instrument, however, the filter is part of the source impedance, so the true source impedance combines R_s, R_f, and C_f. and the Thévenin equivalent source voltage (V_T) is no longer simply V_s. The Thévenin voltage (V_T) and the new output impedance (Z_T) with the filter included become

$$V_T = \frac{V_s}{(R_s + R_f)C_f s + 1} \quad \text{and} \quad Z_T = \frac{R_s + R_f}{(R_s + R_f)C_f s + 1} \tag{58}$$

Figure 26 shows the new equivalent circuit for the cascaded instrument system.

The instrument is now loading this system, and the output (V_o) is equal to the voltage across R_i. Therefore, in terms of the original measured voltage (V_s)

$$\frac{V_o}{V_s} = \frac{1/(1 + (R_s + R_f)/R_i)}{[(1 + R_s/R_f)/(1 + (R_s + R_f)/R_i)]R_f C_f s + 1} = \frac{K_f}{\tau s + 1} \tag{59}$$

where

$$K_f = \frac{1}{1 + (R_s/R_i) + (R_f/R_i)} = \frac{1}{1 + \beta} \tag{60}$$

and

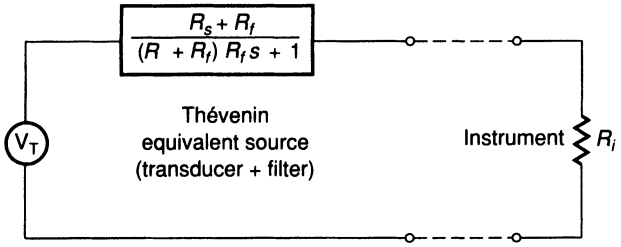


Figure 26 Thévenin equivalent of cascade system.

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$$\tau = \left(\frac{1 + (R_s/R_f)}{1 + (R_s/R_i) + (R_f/R_i)}\right)R_fC_f = \left(\frac{1 + \alpha}{1 + \beta}\right)R_fC_f \tag{61}$$

Our filter has its expected characteristics, $\tau = \tau_f = R_fC_f$ and $K_f = 1$, only if both α and β approach zero, a circumstance that requires $R_s \to 0$ and $R_i \to \infty$ while R_f remains finite. As the instrument resistance (R_i) decreases, the static gain and time constant decrease as well, which means that the break frequency of the filter increases from the designed value because the instrument provides another path for the discharge of the capacitor in the filter. As the transducer resistance (R_s) increases from zero, the static sensitivity (K) again decreases, but this time the break frequency decreases as well.

This analysis shows us that source loading by a filter and filter loading by a readout instrument can cause significant changes in both the designed filter gain and break frequency. Only if α and β in Eqs. (60) and (61) remain equal does the filter break frequency remain unscathed by nonideal source and load impedances. This condition requires that

$$\frac{R_s}{R_f} = \frac{R_s + R_f}{R_i}$$

or

$$R_f^2 + R_sR_f - R_sR_i = 0 \tag{62}$$

Dividing the second of Eqs. (62) by R_s^2 and solving it for R_f/R_s yield

$$\frac{R_f}{R_s} = \frac{1}{2}\left(-1 \pm \sqrt{1 + 4\,\frac{R_i}{R_s}}\right) \cong \sqrt{\frac{R_i}{R_s}} \tag{63}$$

since any realistic measurement system has $R_i/R_s \gg 1$, and since the resistances must be positive, the negative solution is discarded. This approximation is equivalent to saying that R_f should be chosen to be the geometric mean of the estimated or known values of R_i and R_s; that is, $R_f = \sqrt{R_iR_s}$ presuming only that $R_i \gg R_s$. For this choice of filter resistance, the resulting gain is given by

$$K = \frac{1}{1 + \sqrt{R_s/R_i} + R_s/R_i} = \frac{1}{1 + (R_s/R_i)^{3/2}} \cong 1 \qquad \text{for} \qquad R_i \gg R_s \tag{64}$$

If the filter fails these conditions, it must either be carefully designed for the task or it should be an active* filter with a high input impedance and low output impedance.

6.3 Null Instruments

Many instruments are servos, active systems designed to oppose the variable they measure so as to decrease their demand on the complementary energy or power variable to zero. In a steady-state sense, these instruments draw no power or energy from the measurement interface because these energetic requirements are provided by the instrument's power supply. These instruments therefore have infinite or zero input impedance in steady state. Examples abound: Slidewire potentiometers for strain readout and thermocouple readout both measure voltage by servoing to zero current. Servo accelerometers avoid the problems with temperature inherent in the elasticity variations of metals and crystals by servoing the motion of

*Incorporating amplification, usually an operational amplifier.

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the proof mass to zero and measuring the force required to do it. The *differential pressure-cells* used in the process control industry measure differential pressure across a diaphragm or *capsule* while preventing the diaphragm or capsule (separating the two pressures) from moving. They thus avoid having to worry about the nonlinear elastic characteristics of the capsule and seals in the system.* Their output is either the current in a voice coil or the pressure in a bellows necessary to oppose the motion. In all of these cases, however, there is a transient *displacement* until the servo zeros it out, and *some energy must be lost to transfer the information required by the servo in the instrument.* For this reason, null instruments are no better at measuring stored energy, the voltage on a small capacitor for example, than passive but high-impedance instruments, particularly if the energy consumed to reach null is not extremely low.

7 DISTRIBUTED SYSTEMS IN BRIEF

While a detailed study of the input–output relationships for distributed systems is beyond the scope of this chapter, a brief discussion can tie these in to the concepts already covered. All of the systems discussed to this point have been *lumped,* a label that implies physical dimensions are not of importance; the system parts can be considered as *point* objects. Considering a tank, for example, to be a point does not mean that the tank has no dimensions; it merely means that the internal pressure is considered to be the same everywhere within it; conditions in its interior are absolutely uniform. When studying lumped circuits, we are not concerned with the dimensions of the circuit elements or with their distances from each other on the circuit board. In reality, the nodes of the circuit have lengths, but they are ignored for the purposes of analyzing the lumped circuit.

In the mechanical domain, we consider that masses are rigid and behave as if the forces acting were applied at a point, and if we are interested in the distributed properties of an object, we consider only its moment of inertia. In each of these examples, changes in the physical variables of our model are assumed to propagate instantaneously, even though it is well known that all have finite propagation velocities. A system element must be considered to be distributed to have properties that vary with a physical dimension if that assumption is not true. This occurs whenever the dimensions of the object are large compared to the characteristic size of the *events* occurring.

Mechanical disturbances of all kinds propagate at the speed of sound in the medium involved, and electromagnetic disturbances propagate at speeds near the speed of light, depending again on the medium. A hammer blow to the end of a long, slender bar, for example, induces a strain pulse at the struck end which travels into the bar (*informing* the interior of the event) at the speed of sound in compression in the bar material ($c = \sqrt{E/\rho}$, where E is Young's modulus of the material, ρ is its mass density, and c is the propagation speed of compressive or tensile events). If the pulse duration is short, its physical length approaches that of the bar or may be shorter than the bar. Our simple lumped models of the bar's behavior [$F = M\dot{v}$ and $\int F\,dt = Mv(t) - Mv(0)$], which treat it as a solid rigid object, are incorrect. Similarly, if a small explosion, a spark, for example, is initiated in a tank, the pressure in the tank will not remain uniform throughout. Instead, pressure waves will propagate within the tank at the speed of sound until damping takes its toll; we can no longer

*Dry friction would nonetheless be fatal to the instrument because it would be fatal to its servo and would lead to dynamic instabilities of the limit-cycle variety.

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consider the tank to be a simple lumped capacitor, at least in the time scale of the spark event.

7.1 Impedance of a Distributed System

Imagine a long, slender tank of water, a trough, open at the top, and perform the following *thought* experiment. If one end wall were moved inward suddenly, the level of water at that end would rise higher up than was required by the change in tank volume because the remaining water further along the tank would not change level instantaneously. Then, this wave coming down off the tank wall would travel to the other end where it would slosh up the far wall and return. This wave would continue to slosh back and forth at decreasing amplitude as viscosity took its toll, until finally the surface would be calm again at a new, slightly higher level. Each time the wave reached an end, it would be returned in kind, that is, with the same sign.

Now suppose that the far end of the tank opened into a large lake, so large that no level changes would take place when an end wall was moved, and again move the remaining end wall inward quickly. Then, when the first wave reached the opening to the lake, it would leave the tank and be lost in the lake. But that would involve water leaving the tank in excess of the volume change in the tank, and a negative wave would return from the open end to signal the new, lower level required by the loss. The closed end would reflect this wave as a rarefaction, and when that returned to the open end, lake water would spill back in as an upward wave. Eventually these alternating processes would return the level in the tank to that of the lake.

A closed end returns a wave of like kind, and an open end returns a wave of opposite kind. If there are no losses as the waves hit the ends of the tank, a wave of strength $+1$ is reflected with strength $+1$ from a closed end and with a strength -1 from an open end. This implies that there is an end condition somewhere between *closed* and *open* from which a wave will not be reflected at all. A suitably constructed porous wall, in this example, would simply absorb the wave completely by accepting and dissipating all of its energy. The impedance of this wave-matched wall is the wave impedance of the channel and is a characteristic of it, depending on the inductive and capacitive properties of the medium.

The 75- and 300-Ω markings on the antenna connections of a television receiver imply two things: First the input impedances of those terminals are resistive at 75 and 300 Ω, respectively, and, second, the coaxial cable and flat-lead antenna wiring are really waveguides whose wave impedances are 75 and 300 Ω. By matching the impedance of the cable at the receiver terminals, we are assured that all the incoming *wave energy* will be absorbed by the receiver and none will be reflected back up the cable to the antenna and thus lost.

8 CONCLUDING REMARKS

This chapter has demonstrated an alternative viewpoint for the interaction of systems with each other. Control engineers are quite accustomed to transfer functions: relationships in the frequency (s or Laplace) domain between a variable at one point in a system and another at some other point, most often between inputs to and outputs from a controlled system. This chapter has dealt with a special class of these relationships between the complementary variables of power at a single point in a system. These special transfer functions are called driving-point impedances or admittances, and they determine how one subsystem will load or be loaded by another.

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Admittances are the reciprocals of impedances, and both are unique properties of a system. The Laplace operator (s) expresses these properties as a polynomial ratio. The denominator polynomial (when set to zero, it becomes the characteristic equation of the system) is always the same, and the numerator polynomial is a function of the location of the point considered in the system. It was also shown that driving-point impedances are not a function of the controlled variables on any ideal sources the system contains. Instead, all effort sources may be replaced by solid connections, and all flow sources may be removed before the driving-point impedance is computed.

When two systems are connected together at a driving point, port, or pair of terminals, usually so that one can pass energy or information to the other, then there is a favorable relationship between the impedances of the two systems that depends on the objective of the connection. When it is desired to pass energy or power from one system to the other, then the output impedance of the driving system should match the input impedance of the driven system. If neither the driver nor the driven are adjustable, then a transducer, gyrator, or transformer is used to match them by selecting the modulus to achieve the match. Any impedance seen through a transformer, for example, appears to be increased or diminished by the square of the transformer ratio. In a chain of subsystems, it is not necessary to install the transformer at the driving point under consideration; the correct ratio can be determined no matter where it is placed within the chain because the square of the modulus will always appear in one of the driving-point impedances.

If the interconnection represents a measurement interface, then the most favorable relationship between the driving-point impedances is the largest possible mismatch consistent with obtaining the measurement. The ideal instruments for measuring efforts have infinite input impedance and the ideal instruments for measuring flows have infinite input admittances. Instruments that measure the integral of flows, such things as volume, charge, and displacements, should have very low compliance (should displace easily, have low volume themselves, or have small capacitances), while instruments that measure the integral of efforts, such things as flux linkage or momentum, must have low mass or inductance.

The operating point of a pair of coupled systems is at the intersection of their input and output characteristics in the power or energy plane. If one of these, for example the source or output characteristic, exists in the power plane, that is, is static, but the other is energetic (i.e., dynamic: massive, inductive, capacitive, etc.), then the source characteristic must enclose the trajectory of the load characteristic at the highest frequency of interest, and ideally, the source characteristic and load trajectory should be tangent at the maximum power point or should be made tangent there by suitable choice of system parameters.

The key issue in this chapter is this: Whenever two dynamic systems are connected, an interaction occurs. If the connection is to meet its objectives, the nature of this interaction must be explored and controlled.

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